Non-Clausal Redundancy Properties

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logics
LOGICAL METHODS IN
COMPUTER SCIENCE

Motivation

- SAT solvers are used where correctness matters
  - Verifying hardware and software [CBRZ01, GPB01, KSHK07]
  - Subroutines in other reasoning tools [BSST21, Vor14]
  - Search for solutions to math problems [HKM16, KL15]
- Solvers should produce externally-checkable certificates
  - **Example**: if F is UNSAT, produce a resolution refutation
- Most modern proof systems infer redundant clauses

Clause C is **redundant** w.r.t. formula F if $F \equiv_{\text{SAT}} F \wedge C$

- Examples: RUP [GN03], RAT [WHH14], PR [HKB17], ...

Motivation

- PR has short refutations for many **hard** problems (see [BT19])
  - Problems with no polynomial-length resolution refutation
- CDCL searches for resolution refutations [BKS04]
- PR presents the potential for major speed-ups in solving
  - Not obvious how to exploit this in practice (but see SDCL [HKB19])
- Established techniques already offer speed-up beyond CDCL

	`--gauss`	`--no-gauss`
`urquhart_s5_b3`	0.1 sec	$>$ 10 hrs

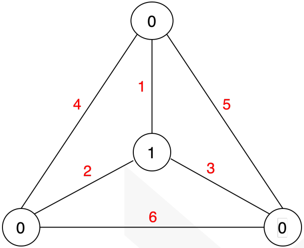
- Clausal proof systems struggle to express non-clausal reasoning
  - Typically disable XOR, cardinality reasoning if proofs required

* Want proofs that express these techniques as well

A hard problem

- Graph G, each v has a **charge** $\gamma(v) \in \{0, 1\}$, total charge is odd
- Variable $\boldsymbol{x_e}$ for each edge e in G
- Formula $\boldsymbol{F_{G,\gamma}}$: for each v, parity of the incident x_e equals $\gamma(v)$

```
p cnf 6 16
1 2 3 0
1 -2 -3 0
-1 2 -3 0
-1 -2 3 0
1 4 -5 0
1 -4 5 0
-1 4 5 0
-1 -4 -5 0
2 4 -6 0
2 -4 6 0
-2 4 6 0
-2 -4 -6 0
3 5 -6 0
3 -5 6 0
-3 5 6 0
-3 -5 -6 0
```



- For certain graphs, no short resolution proofs [Tse70, Urq95]
- **Tseitin formulas** have short DRAT, (D)PR proofs [BT19, HKB19]

Redundancy Properties

$$P(F, C) \Rightarrow C \text{ is redundant w.r.t. } F$$

RUP	unit propagation on $F \wedge \neg C$ produces conflict: $\boldsymbol{F \vdash_1 C}$
RAT	for some $\ell \in C$ the clause $C \vee D$ is RUP for all $D \vee \neg \ell \in F$
PR	for some assignment ω, both $C\|_\omega = \top$ and $F\|_{\neg C} \vdash_1 F\|_\omega$

- Add clauses to F that meet the redundancy property
- Prove "UNSAT" by eventually adding the empty clause $\perp$
- Deciding whether a clause is PR is NP-complete [HKSB17]

XOR Reasoning

- Can Tseitin formulas be solved without looking for PR clauses?
- CryptoMiniSAT [SNC09], Lingeling [Bie18], . . . use **XOR reasoning**

```
p cnf 6 16
1 2 3 0          2 4 -6 0
1 -2 -3 0        2 -4 6 0
-1 2 -3 0        -2 4 6 0
-1 -2 3 0        -2 -4 -6 0
1 4 -5 0         3 5 -6 0
1 -4 5 0         3 -5 6 0
-1 4 5 0         -3 5 6 0
-1 -4 -5 0       -3 -5 -6 0
```

$$x_1 \oplus x_2 \oplus x_3 = 1$$
$$x_1 \oplus x_4 \oplus x_5 = 0$$
$$x_2 \oplus x_4 \oplus x_6 = 0$$
$$x_3 \oplus x_5 \oplus x_6 = 0$$

- Extract XOR constraints, solve efficiently with **Gaussian elimination**
- But expressing this is a challenge for RUP, RAT, PR, ...
- Not just XORs. Example: reasoning about **cardinality**

Non-clausal Redundancy

Function g is **redundant** w.r.t. function f if $f \equiv_{\mathsf{SAT}} f \wedge g$

Want non-clausal redundancy properties, proof systems

- Efficiently-checkable
- Easily express existing solver techniques
- Extend existing proof systems

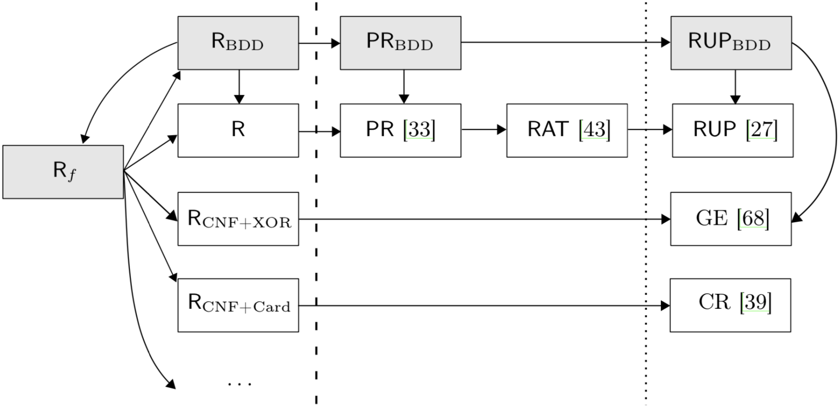
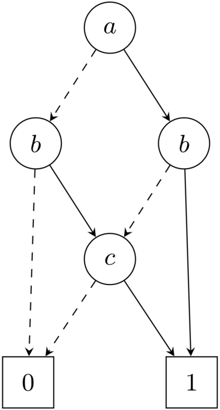


Fig. 1: Different notions of redundancy and their relationships. An arrow from A to B indicates A generalizes B. Properties to the right of the thick dashed line are polynomially checkable; those to the right of the thin dotted line only derive logical consequences. Novel properties defined in this paper are grey.

Binary Decision Diagrams

- BDDs [Ake78, Bry86, Lee59] compactly express Boolean functions
- Long history in SAT (e.g. [BH21, DK03, FKS+04, MM02, PV04])



- **Shannon decomposition**

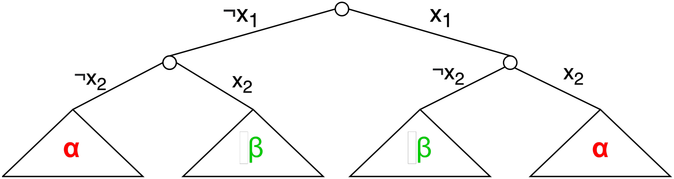
$$f = (\neg x \wedge f|_{\neg x}) \vee (x \wedge f|_{x})$$

- $a + b + c \geq 2$
- Clauses are easy to represent
- Formulas in general are not
- Conjunction of BDDs:

$$F = f_1 \wedge \cdots \wedge f_n$$

Redundancy for BDDs

- $x_1 \oplus x_2 = 1$ is redundant w.r.t. $F \iff$ not all F-solutions are in $\boldsymbol{\alpha}$



- Want a function $F|_\alpha$ such that
  - If assignment τ is in α then $F|_\alpha(\tau) = F(\tau)$
  - $F|_\alpha$ is simpler than F
- A **generalized cofactor** of F by $x_1 \oplus x_2 = 0$

Generalized Cofactor

- Can compute $f|_g$ using **constrain** operation [CM90, TSL+90]

Constrain(f, g), for $g \neq 0$, produces the BDD $f \circ \pi_g$, with π_g given by

$$\pi_g(\tau) = \begin{cases} \tau & \text{if } g(\tau) = 1 \\ \underset{\{\tau' \mid g(\tau')=1\}}{\arg\min} \; d(\tau, \tau') & \text{otherwise} \end{cases}$$

where $d(\tau, \tau') = \sum_{i=1}^{n} |\tau(x_i) - \tau'(x_i)| \cdot 2^{n-i}$ for variables $x_1 \prec \cdots \prec x_n$.

- Usually smaller than f and can be computed efficiently
- Distributes over $\wedge$, so $(f_1 \wedge \cdots \wedge f_n)|_g = f_1|_g \wedge \cdots \wedge f_n|_g$

Redundancy for BDDs

Can characterize redundancy as follows:

A BDD g is redundant w.r.t. $f_1 \wedge \cdots \wedge f_n$ iff there exists $\omega = \ell_1 \wedge \cdots \wedge \ell_k$:

1. $g|_\omega$ is the "always-true" BDD 1
2. $f_1|_{\neg g} \wedge \cdots \wedge f_n|_{\neg g} \models f_1|_\omega \wedge \cdots \wedge f_n|_\omega$

- Compare with the following characterization for clauses [HKB20]

A clause C is redundant w.r.t. formula F iff there exists an assignment ω:

1. ω satisfies C
2. $F|_{\neg C} \models F|_\omega$

- Can check redundancy by checking this implication
- Make this efficient by using a restricted implication relation
- For clause redundancy properties, use RUP

BDD Unit Propagation

UnitProp$(f_1, \ldots, f_n)$

repeat

  if $f_i = 0$ **or** $f_i = \neg f_j$ **for some** $1 \leq i, j \leq n$ **then**

    return "conflict"

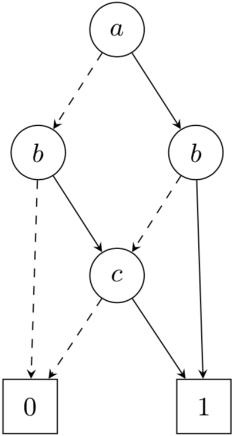
  if $U(f_i) \neq \emptyset$ **for some** $1 \leq i \leq n$ **then**

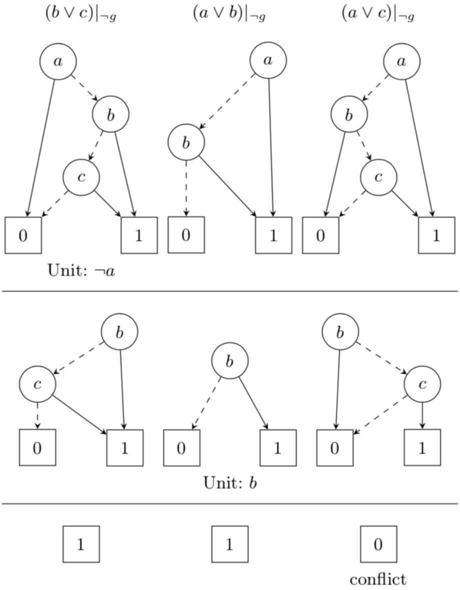
    $f_j := f_j|_{\bigwedge U(f_i)}$ **for all** $1 \leq j \leq n$

until no update to $f_1, \ldots, f_n$

- $U(f)$ = set of literals implied by the BDD f
- Propagates these implied literals through the collection
- If UnitProp$(f_1|_{\neg g}, \ldots, f_n|_{\neg g})$ = "conflict" then $f_1 \wedge \cdots \wedge f_n \models g$

Reverse UnitProp Example





Redundancy Properties for BDDs

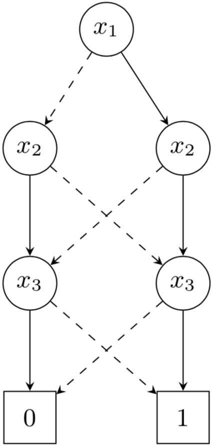
A BDD g is **RUP$_{\text{BDD}}$** w.r.t $f_1 \wedge \cdots \wedge f_n$ if
* UnitProp$(f_1|_{\neg g}, \ldots, f_n|_{\neg g})$ = "conflict", or
* UnitProp$(f_1|_p, \ldots, f_n|_p)$ = "conflict" for each 0-path p in g (**RUP$_{\text{path}}$**)

- Efficiently decidable, generalizes clausal RUP

A BDD g is **PR$_{\text{BDD}}$** w.r.t $f_1 \wedge \cdots \wedge f_n$ if for some ω
* $g|_\omega = 1$, and
* for all $1 \leq i \leq n$ either $f_i|_\omega = 1$ or $f_i|_\omega$ is RUP$_{\text{BDD}}$

- Efficiently checkable, given ω
- Generalizes clausal PR property

Gaussian Elimination with RUP_{BDD}

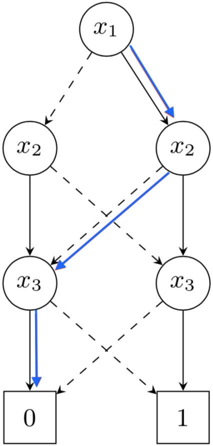


$x_1 \oplus x_2 \oplus x_3 = 1$

```
p cnf 6 16
1 2 3 0            2 4 -6 0
1 -2 -3 0          2 -4 6 0
-1 2 -3 0          -2 4 6 0
-1 -2 3 0          -2 -4 -6 0
1 4 -5 0           3 5 -6 0
1 -4 5 0           3 -5 6 0
-1 4 5 0           -3 5 6 0
-1 -4 -5 0         -3 -5 -6 0
```

- If $f_1 \wedge \cdots \wedge f_n$ includes the CNF of an XOR, the BDD is RUP_{path}

Gaussian Elimination with RUP_{BDD}



$p = x_1 \wedge \neg x_2 \wedge x_3$

```
p cnf 6 16
1 2 3 0
1 -2 -3 0
-1 2 -3 0
-1 -2 3 0
1 4 -5 0
1 -4 5 0
-1 4 5 0
-1 -4 -5 0
2 4 -6 0
2 -4 6 0
-2 4 6 0
-2 -4 -6 0
3 5 -6 0
3 -5 6 0
-3 5 6 0
-3 -5 -6 0
```

- If $f_1 \wedge \cdots \wedge f_n$ includes the CNF of an XOR, the BDD is RUP_{path}

Gaussian Elimination with RUP_{BDD}

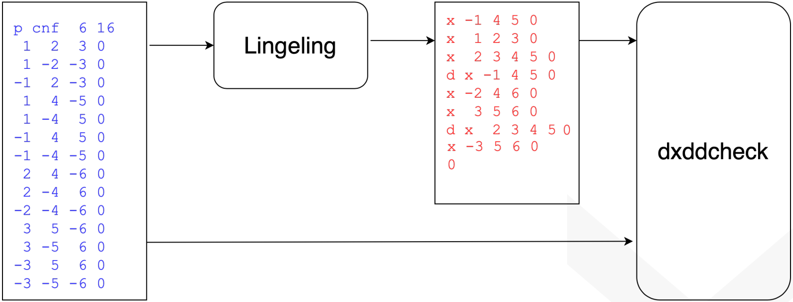
- From constraints X and Y, want to infer $X \oplus Y$

$$\begin{array}{rl} 1 \oplus 2 \oplus 3 & = 1 \\ 1 \oplus 4 \oplus 5 & = 0 \\ \hline 2 \oplus 3 \oplus 4 \oplus 5 & = 1 \end{array}$$

- We show $X|_{\neg(X \oplus Y)} = \neg Y|_{\neg(X \oplus Y)}$, so $X \oplus Y$ is RUP_{BDD}
- Proof system using RUP_{BDD} or PR_{BDD}
  - Easily expresses Gaussian elimination steps
  - Extends corresponding clausal property

Results

- **dxddcheck**: prototype implementation in Python
- Checks proofs in this subsystem capturing Gaussian elim.
  - Allows clause and XOR addition



- Proofs can be extracted from Lingeling output
- dxddcheck checks that the XOR constraint is redundant

Results

Formula	number of variables	number of clauses	solving time (s)	proof lines	proof size (KB)	checking time (s)
rpar_50	148	394	0.1	297	7	0.34
rpar_100	298	794	0.1	597	15	1.35
rpar_200	598	1594	0.2	1197	35	6.67
mchess_19	680	2291	0.0	1077	41	4.07
mchess_21	836	2827	0.1	1317	50	5.09
mchess_23	1008	3419	0.1	1581	63	6.42
urquhart-s5-b2	107	742	0.0	150	7	0.95
urquhart-s5-b3	121	1116	0.1	150	9	1.64
urquhart-s5-b4	114	888	0.0	150	8	1.20

- urquhart benchmarks: Tseitin formulas
- With Gaussian elim, Lingeling solves all almost instantly
- Without Gaussian elim, Lingeling and Kissat timeout
  - Only rpar_50 was solved in $<$ 10 hours
  - Proof in this case was $\approx$ 6911 MB

Conclusion

- Generalized redundancy beyond clauses
- Define properties, proof systems based on redundant BDDs
- Proof systems easily express Gaussian elimination
- Prototype results confirm this approach is practical

Future work: cardinality reasoning, (certified) $\mathrm{PR}_{\mathrm{BDD}}$ proof checker

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